Eigenvalues and Isomorphism

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The moral of the story is this: Eigenvalues/vectors are about *stable subspaces* while kernel and image (invertibility) are about *permuting and scaling subspaces.*

For these notes we will consider matrices over the complex numbers.

Lemma. *All eigenvalues are non-zero iff the matrix is invertible.*

Proof. Put the matrix in JNF, the eigenvalues are the diagonal entries. The JNF is uppertriangular and hence its determinant is just the product of the eigenvalues. Hence the matrix has a non-zero determinant (invertible) iff all its eigenvalues are non-zero.

Lemma. *We have an inclusion eigenspaces for eigenvalues* $\neq 0 \subseteq \mathrm{Im}(f)$ *where* f *is some linear transformation* $V \to V$*.*

Proof. This is clear since if φ is an eigenvector for eigenvalue λ then $f(\frac{1}{\lambda}\varphi) = v$.

I also want to give an example where this inclusion is strict. Consider a non-trivial Jordan block

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then the only eigenvalue is 1 and the eigenspace is elements of the form $(a, 0)$ but this map is surjective, in other words full rank. Hence eigenspace $\subset$ Im. This is a general feature of such Jordan blocks, as they appear when the eigenspace is a lower dimension that the eigenvalues multiplicity.

Lemma. *We have an equality* $\ker(f) =$ *eigenspace of eigenvalue* 0*.*

Proof. $f(X) = 0X = 0$ iff X is an eigenvector for eigenvalue 0.

This gives the decomposition of V into

$$V = \ker A \oplus \mathrm{Im}A = \text{0-eigenspace} \oplus \text{non-zero-eigenspace} \oplus \text{Jordan blocks}$$

or alternitively we can see that

$$\mathrm{Im}A \cong \text{non-zero-eigenspace } \textbf{with multiplicity}$$

Although this is an isomorphism now and no longer equality.